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COMMENT

On a magnetic analogy of the de Gennes' ant in a labyrinth

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Abstract. We present a rigorous proof that the discrete Laplace transforms of the displacement probabilities of the myopic and the blind ant are proportional to the spin–spin correlation functions of a system of continuous spins in a random crystal field. From this it follows that one can use the Hamiltonian of the spin system to cacluahe, for instance, the fractal properties of the paths of the ants in a random medium, and in particular one can demonstrate in a simple way that the asymptotic properties of the two ants are equivalent.

The problem of diffusion in inhomogeneous media has been studied by many investigators since de Gennes (1976) introduced the picture of the ant in a labyrinth. The ant performs a random walk, and the research problem consists in determining asymptotic properties of the walk when the number of completed steps N tends to infinity. Numerical investigations of the problem (Majid *et al* 1984, Seifert and Suessenbach 1984) suggested that the two types of ants, namely the myopic and the blind ant (Mitescu and Roussenq 1983), might have the same asymptotic behaviour. Although recently this suggestion has been analytically vindicated (Harris *et al* 1987, Maritan 1988), nevertheless it should be useful to have a different point of view on the same problem.

Ever since it was recognised that there is a remarkable correspondence between the self-avoiding random walk model and the n-component spin model in the limit $n \to 0$ (de Gennes 1972, Domb 1972, des Cloizeaux 1974), there has been a noticeable endeavour to map the statistics of various random walks to the statistical mechanics of the spin Hamiltonian systems (see, for instance, Christou and Harris 1987). Here we demonstrate that asymptotic properties of the myopic and the blind ant problem can be mapped to the critical behaviour of a system of interacting continuous spins in a random crystal field. This mapping enables one to attack the problem by the renormalisation group (RG) method and, in a simple way, points out the equivalence of geometrical critical properties of infinite paths of the two ants.

The above-mentioned mapping can be effected by proving that the mean displacement of the ants for large N is proportional to the correlation length of the spin system model near its critical point in an appropriate crystal field. In what follows it is useful first to present explicitly certain properties of statistics of random walks in random media. Specifically, we first introduce the matrix element $P_{xx'}(N)$ which is equal to the probability of finding the ant (myopic, blind or any other) at the site x, after N

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steps, providing it has started its walk at the site x'. By forming the sum $\Sigma\,(x-x')^2 P_{xx'}(N)$ over all x, and taking its average over all initial sites x', one obtains the mean squared end-to-end distance $\langle R_N^2\rangle$, which for very large N behaves according to the power law

$$\langle R_N^2\rangle \sim N^{2/d_w} \tag{1}$$

where d_w is the fractal dimension of the random walker path.

It is convenient to study the discrete Laplace transforms of the quantities introduced above, i.e.

$$\mathscr{P}_{xx'}(\omega) \equiv \sum_{n=0}^{\infty} \frac{P_{xx'}(N)}{(1+\omega)^{n+1}} \qquad \omega \geqslant 0 \tag{2}$$

and

$$R^2(\omega) \equiv \omega \sum_x (x-x')^2 \mathscr{P}_{xx'}(\omega). \tag{3}$$

Of course, in the further analysis we will consider the quantity $\langle R^2(\omega)\rangle$ which is the average of (3) over all possible initial sites x'. It is easy to show that, from relation (1) and from the definitions (2) and (3), there emerges the singular behaviour

$$\langle R^2(\omega)\rangle \sim \omega^{-2/d_w} \tag{4}$$

for $\omega \approx 0$.

Now we return to the matrix $\hat{P}(N)$, defined by the elements $P_{xx'}(N)$, and introduce the new matrix $\hat{L}$ defined by

$$L_{xy} = \begin{cases} w_{xy} & \langle x, y\rangle \\ 1-\sum\limits_x w_{xy} & x=y \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where w_{xy} is a probability for a random walker to skip from the site y to one of its nearest-neighbouring sites x. Here we have used the standard notation $\langle x, y\rangle$ to specify sites which are nearest neighbours (NN). For Markovian processes, defined by w_{xy}, it is not difficult to verify the following relation:

$$\hat{P}(N) = \hat{L}^N. \tag{6}$$

Besides, formulae (2) and (6) imply the additional matrix relation:

$$\hat{\mathscr{P}}(\omega) = [(1+\omega)\hat{I} - \hat{L}]^{-1} \tag{7}$$

where elements of $\hat{\mathscr{P}}(\omega)$ are given by (2) and $\hat{I}$ is the unit matrix.

Finally, we recall the definition of the probability

$$w_{xy} = \frac{w}{z} \qquad 0 < w \leqslant 1 \tag{8a}$$

for the blind ant and

$$w_{xy} = \frac{w}{z_y} \qquad 0 < w \leqslant 1 \tag{8b}$$

for the myopic ant. Here z_y is the number of the NN for the site y, whereas z is the maximum coordination number within the random medium studied.

At this point we introduce, in two steps, the spin Hamiltonian which will be mapped to the random walk problem. First, we recall the Blume-Emery-Griffiths Hamiltonian (Blume *et al* 1971)

$$\mathcal{H} = -J \sum_{\langle x,y \rangle} s_x s_y + \Delta \sum_x s_x^2 - H \sum_x s_x \qquad s_x = 0, \pm 1 \tag{9}$$

where J is the exchange interaction parameter, s_x is the discrete spin variable, Δ is the crystal field parameter and H is the magnetic field. Next, we allow s_x to be the continuous variable ($-\infty < s_x < +\infty$) and let H equal zero. Besides, we assume that Δ is a function of the site x. The new Hamiltonian is of the Gaussian type and can be written in the matrix form:

$$\mathcal{H} = \mathbb{S}^{\mathrm{T}} \hat{M} \mathbb{S} \tag{10}$$

where $\mathbb{S}$ is the column vector whose elements are the continuous spin variable s_x, $\mathbb{S}^{\mathrm{T}}$ is the transposed vector $\mathbb{S}$ and the elements of the matrix $\hat{M}$ are given by

$$M_{xy} = \begin{cases} 0 & x \neq y \text{ and } (x, y) \text{ are not NN} \\ \Delta_x \delta_{xy} - J/2 & \text{otherwise} \end{cases} \tag{11}$$

with δ_{xy} being the Kronecker delta function.

We are going to perform the mapping of the random walk problem to the above Hamiltonian by calculating the corresponding spin-spin correlation function:

$$K_{xy} = \frac{1}{Z} \int s_x s_y \exp(-\beta \mathcal{H}) \prod_x \mathrm{d}s_x \tag{12a}$$

where Z is the partition function:

$$Z = \int \exp(-\beta \mathcal{H}) \prod_x \mathrm{d}s_x \tag{12b}$$

with β being the reciprocal of the product of the Boltzmann constant and temperature T. The correlation function (12a) can be exactly calculated by noticing that the matrix $\hat{M}$ defined by (11) is a symmetric matrix. Such a matrix can be diagonalised, and thereby we obtain

$$\hat{K} = \frac{1}{2\beta} \hat{M}^{-1} \tag{13}$$

where $\hat{K}$ is the matrix defined by (12).

If we now return to the relation (7) and compare it with the matrix (11), we can see that the following relation is valid:

$$(\hat{\mathcal{P}}^{-1}(\omega))_{xy} = \frac{2w_{xy}}{J} M_{xy} \qquad x \neq y. \tag{14}$$

It should be emphasised that we can choose the crystal field parameter Δ_y in such a way that the above relation holds for $x = y$ as well. The requisite choice is

$$\Delta_y = \frac{J(\hat{\mathcal{P}}^{-1}(\omega))_{yy}}{2w_{yy}} \tag{15}$$

where w_{yy} has to be formally defined by $w_{yy} = w_{xy}$, which is possible since in the studied case (8) the probability w_{xy} does not depend on the first index x. According to (7), the above relation (15) obtains the specific form

$$\Delta_y = \tfrac{1}{2} z_y J (1 + z\omega / z_y w) \tag{16a}$$

for the blind ant and

$$\Delta_y = \tfrac{1}{2} z_y J (1 + \omega / w) \tag{16b}$$

for the myopic ant.

One can directly verify that our findings (13)–(15) imply the matrix element relation:

$$\frac{w_{xy}}{\beta J} (\hat{K}^{-1})_{xy} = (\hat{\mathcal{P}}^{-1})_{xy} \tag{17}$$

and, since w_{xy} does not depend on the index x, the following relation is also true:

$$\mathcal{P}_{xy}(\omega) = \frac{\beta J}{w_{xy}} K_{xy}. \tag{18}$$

With this formula we have completed the necessary set of relations which facilitates comparison of the criticalities of the random walks and the spin system.

First, we accept the standard scaling form of the spin-spin correlation function:

$$K_{xy} = \frac{1}{|x-y|^a} F\left(\frac{|x-y|}{\xi}\right) \tag{19}$$

where a is a non-negative constant, F is a one-variable function and ξ is the correlation length. Our finding (18), together with the assumed form (19), implies

$$\mathcal{P}_{xy}(\omega) = \frac{\beta J}{w_{xy}} \frac{1}{|x-y|^a} F\left(\frac{|x-y|}{\xi}\right). \tag{20}$$

Next, inserting (20) into formula (3) and taking appropriate averages over all possible initial sites should result in

$$\langle R^2(\omega)\rangle^{1/2} \sim \xi \tag{21}$$

as the only parameter with the dimension of length that appears in the distribution (20) is the correlation length ξ. The above formula, together with relation (4), provides

$$\xi \sim \omega^{-1/d_w} \qquad \omega \approx 0. \tag{22}$$

Therefore we can see that the critical parameter for the spin system is the crystal field parameter, which is, in general, related to ω by (15), and in the particular case of the blind and myopic ants, by (16). In other words, relation (22) implies that one can calculate the critical exponent $1/d_w$ of the random walk by calculating the critical exponent of the spin system correlation length ξ.

As regards the de Gennes' ants, knowing from (22) that the critical behaviour appers at $\omega = 0$, we can see from relations (16) that the critical point for both the blind and myopic ant is at the same value $\Delta_y = z_y J/2$. This inference is independent of the specific random medium studied, and within the RG treatment of the relevant spin Hamiltonian it means there exists a unique fixed point which describes criticalities of both ants. Hence, it transpires that, in any random medium, the two ants cannot have different d_w, which is in accord with the arguments presented by Harris *et al* (1987) and Maritan (1988).

In conclusion, we see that there is a close analogy between the behaviour of the random walks in the limit $N \to \infty$ ($\omega \to 0$) and the criticality of a continuous spin system in the random crystal field (15). This analogy can be exploited to calculate various asymptotic properties of random walks. Besides, it can be used, for instance, to study dynamic characteristics of finitely ramified fractals. In the case of these fractals, the spin Hamiltonian can be treated exactly by the RG method. Thus, one can find exact values of d_w for such fractals, and thereby the corresponding spectral dimensions d_s via the relation $d_s = 2d_f/d_w$ (Alexander and Orbach 1982), where d_f is the fractal dimension.

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References

Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
Blume M, Emery V I and Griffiths R 1971 *Phys. Rev.* A **4** 1071
Christou A and Harris A B 1987 *Time-Dependent Effects in Disordered Materials* ed R Pynn and T Riste (New York: Plenum) p 481
de Gennes P G 1972 *Phys. Lett.* **38A** 339
—— 1976 *La Recherche* **7** 919
des Cloizeaux J 1974 *Phys. Rev.* A **10** 1665
Domb C 1972 *J. Phys. C: Solid State Phys.* **5** 1399
Harris A B, Meir Y and Aharony A 1987 *Phys. Rev. B* **36** 8752
Majid I, Ben-Avraham D, Havlin S and Stanley H E 1984 *Phys. Rev.* B **30** 1626
Maritan A 1988 *J. Phys. A: Math. Gen.* **21** 859
Mitescu C and Roussenq J 1983 *Percolation Structures and Processes* (*Ann. Israel Phys. Soc.* **5**) ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 81
Seifert E and Suessenbach M 1984 *J. Phys. A: Math. Gen.* **17** L709